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Nonlinear Dynamics and Chaos: Application to Financial Markets in Pakistan*

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1. INTRODUCTION

Recently there has been an increased interest in the theory of chaos by macroeconomists and financial economists. Originating in the natural sciences, applications of the theory have spread through various fields including brain research, optics, metereology, and economics. The attractiveness of chaotic dynamics is its ability to generate large movements which appear to be random, with greater frequency than linear models.

Two of the most striking features of any macro-economic data are its random-like appearance and its seemingly cyclical character. Cycles in economic data have often been noticed, from short-run business cycles, to 50 years Kodratiev waves. There have been many attempts to explain them, e.g. Lucas (1975), who argues that random shocks combined with various lags can give rise to phenomena which have the appearance of cycles, and Samuelson (1939) who uses the familiar multiplier accelerator model. The advantage of using non-linear difference (or differential) equation models to explain the business cycle is that it does not have to rely on *ad hoc* unexplained exogenous random shocks.

In the early 1970s the geometric random walk commanded great respect as a description of asset pricing. Indeed, the stylised fact about stock prices was that they behave like random walks. The equilibrium asset pricing models that followed the work of Rubinstein (1976); Lucas (1978) and others, by linking

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stock returns to consumption variability provided, in principle, a role for nonlinearities. However, the attempts to implement these models involved parameterisation where, in the absence of external random shocks, fluctuations would be absent. Nonlinear models are, again, an attractive alternative to explain stock price fluctuations.

As there are a number of different definitions of chaos in use (positive topological entropy, positive Liapunov exponents, existence of a strange attractor, etc.) and different types of chaos (ergodic chaos, topological chaos), it becomes difficult to rigorously define chaos without using a lot of technical terms which is not the purpose of this paper. At an informal level chaos is a nonlinear deterministic system which is both sensitive to initial conditions and has a periodic motion. It is a process which is able to produce motions so complex that they appear completely random. Brock (1986) and Majumdar and Mitra (1994) provide exact mathematical definitions aimed at the economics and finance profession.

A substantial amount of recent research has sought to elucidate the role of nonlinearity and chaos in macroeconomic models. Some of the work has been theoretical, attempting to ascertain whether simple nonlinear deterministic models can exhibit the kind of fluctuations typically found in economic data. Majumdar and Mitra (1994); Baumol and Benhabib (1989); Kelsey (1988) and Scheinkman (1990) have surveys of economic models, particularly growth models, which produce chaotic behaviour. Other work has been empirical, and tests for the possibility that actual economic and financial time series are characterised by chaotic dynamics. LeBaron (1991) provides a survey of the empirical work.[1] Both lines of research are considered to be in the early stages.

The purpose of this paper is to extend the empirical work on chaotic dynamics to financial markets in a developing country. We investigate whether the returns implied by the State Bank's weekly stock price indices contain any nonlinearities or chaos. Clearly if stock returns are governed by a chaotic process, it should have short term predictability. However, traditional linear forecasting methods would not work in that case and thus would be inappropriate. The paper proceeds as follows. The next section describes the method used to test for nonlinear dependence. The results of applying the test to stock returns in

[1]There are several recent working papers by LeBaron, Scheinkman, Hsieh, Brock and others that use, extend and refine tests for nonlinearity. To stay abreast of the rapidly expanding literature in nonlinear dynamics, the interested reader may want to correspond with them directly. The lag between submission to journals and eventual publication means that much of the work appearing in journals was done sometime ago and is thus perhaps dated.

Pakistan are reported in Section 3. Section 4 concludes the paper by highlighting the main findings, and suggestions for further research.

2. TESTING NONLINEARITY

At present methods to distinguish whether a time series has been generated by a stable linear/nonlinear stochastic system or a deterministic nonlinear system giving rise to chaotic dynamics are still in their infancy. While many tests have been developed to detect nonlinear dependence, including tests for chaos, the BDS test Brock, Dechert and Scheinkman (1987) has gained widespread acceptance because of its ability to identify nonlinear dependence in economic and financial data.[2]

Early efforts in uncovering nonlinear dependence in economic and financial data consisted simply of using certain tools developed in the mathematics and physics literature. Prominent among them was the use of the correlation dimension developed by Grassberger and Procaccia (1983). Let the ordered sequence $\{X_t\}$, $t = 1, \ldots, N$, represent the observed time series. The correlation integral is defined as:

$$C(\varepsilon) = \lim_{T\to\infty}[\, 2/T\,(T-1)\Sigma I_{\varepsilon}\,(\varepsilon - |X_i - X_j\,|\,],\quad i < j \qquad \ldots \quad \ldots \quad \ldots \quad (1)$$

where I_{ε} is an indicator function that equals one if $|X_i - X_j| < \varepsilon$ and zero otherwise. The correlation integral $C(\varepsilon)$ measures the fraction of the total number of pair of points of $\{X_t\}$ that are within a distance of ε from each other. The correlation integral is used by Grassberger and Procaccia to define the correlation dimension of $\{X_t\}$:

$$CD = \lim_{\varepsilon\to 0}\,[\log C\,(\varepsilon)/\log\,\varepsilon\,],\ \text{if the limit exists} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

While the estimation of the correlation dimension is straightforward and has been applied by physicists, its application to economic and financial data raises a number of issues. These issues are discussed at length in Hsieh (1991). First the time series in economics and finance tend to be much shorter than it seems necessary to obtain good estimates of CD. As perspective on the order

[2]Some popular examples are in Engle (1982); Hinich (1982) and Tsay (1986).

[3]An attractor is the set of points of the path that represents the long term behaviour of the dynamical system.

of magnitudes involved, scientists typically use 100,000 or more data points to detect, at most, low dimensional chaotic systems. Second, as Ramsey and Yuan (1989) show, the CD is biased downward in data sets even with as many as 2,000 observations. Third there is no statistical theory regarding the sampling distribution of the sample correlation dimension.

To deal with the problems associated with using the correlation dimension, Brock, Dechert and Scheinkman (1987) devised a statistical test based on the correlation integral given in Equation 1. Given $\{X_t\}$ form n-histories of it. These are denoted as follows:

$$\text{1–history:} \quad X_t^1 .$$

$$\text{2–history:} \quad X_t^2 = (X_{t-1}, X_t)$$

$$\vdots \qquad \vdots$$

$$n\text{–history:} \quad X_t^n = (X_{t-n+1}, \ldots, X_t) \qquad \ldots \quad (3)$$

An n-history is a point in n-dimensional space and is called "embedding dimension". Calculate the correlation integral $C_N(\varepsilon)$. This is interpreted as the fraction of the n-histories that are within ε of each other. Brock, Dechert and Scheinkman show that under the null hypothesis $\{X_t\}$ is independently and identically distributed (iid from now) with a nondegenerate density F, $C_N(\varepsilon) \to C_1(\varepsilon)^N$ with probability one, as $t \to \infty$, for any fixed N and ε. Furthermore, they show that $\sqrt{t}\,[\hat{C}_N(\varepsilon) - C(\varepsilon)^N]$ has a normal limiting distribution with zero mean and a finite variance $\sigma_N^2(\varepsilon)$.[4] Therefore, under the null hypothesis the BDS statistic

$$BDS = \sqrt{t}\,[C_N(\varepsilon) - C(\varepsilon)^N]/\sigma_N(\varepsilon) \qquad \ldots \quad (4)$$

has a standard normal limiting distribution. The BDS statistic gives some information about the type of dependence in the data. Suppose BDS is a positive number. The probability of any two N-histories being close together is higher than the Nth

[4]Formulas for estimators of variances are provided in Scheinkman and LeBaron (1989).

power of the probability of any two points, X_i and X_j, being close together. This implies that some patterns of stock prices occur more frequently than would be predicted had the data been truly random. Hsieh (1989, 1991) provides Monte Carlo evidence that the BDS statistic for foreign exchange rates and stock prices can reliably be approximated by its asymptotic distribution. Simulations reported in Brock, Dechert and Scheinkman (1987); Brock *et al.* (1988) and Hsieh (1991) show that it has good power against many of the favourite nonlinear alternatives.

It is important to note that the BDS statistic tests the null hypothesis of a random independent and identically distributed (iid) system. A rejection of the null hypothesis is consistent with some type of dependence in the data which could result from a linear stochastic system (e.g. ARMA processes), a nonlinear stochastic system (such as ARCH/GARCH processes), or a nonlinear deterministic system which could be low order chaos. Therefore, it is important to remove all linear influences from the data before employing the BDS statistic to test for nonlinearity.

3. APPLICATION TO PAKISTAN STOCK RETURNS

The data used for this paper include weekly stock returns for the period July 1986 to June 1992 for a total of 310 observations. The returns are calculated as logarithmic first differences of the State Bank General Index of Share Prices in local currency.

In total, eleven series of returns are examined for nonlinearity. The first series represents SPB's overall index of share prices, which is a value weighted and broadly based index. The other ten series represent general indices of share prices of specific industrial groups. These include share prices of firms producing the following products: (1) Cotton and Other Textiles, (2) Chemicals, (3) Engineering, (4) Sugar and Allied Industries, (5) Paper and Board, (6) Cement, (7) Fuel and Energy, (8) Transport and Communication, (9) Insurance and Finance, and (10) Miscellaneous Industries selling tobacco, jute, and vanaspati and allied products.

Table 1 provides summary statistics of the data. All weekly mean returns (column 1) are positive and statistically different from zero for two-tailed test at the 10 percent or less levels of significance. Annual returns accruing to the different stock groups can be inferred from the weekly mean returns. For example, the weekly compounded average rate of return for the general stock index amounts to 29 percent annually.

Table 1

Summary Statistics of Stock Price Indices, 1987–93
$\log (S_t / S_{t-1}).100$

Index	Mean (*t*-value)	St. Dev.	Skewness	Kurtosis	Minimum	Maximum
General	0.49 (5.50)	1.56	1.21	4.72	–4.09	8.19
Cotton	0.52 (4.48)	2.03	1.12	3.02	–4.34	10.95
Chemicals	0.45 (4.54)	1.75	0.65	1.22	–4.98	6.45
Engineering	0.35 (3.46)	1.80	0.57	2.03	–5.10	7.70
Sugar	0.38 (2.88)	2.30	0.38	13.28	–14.65	15.50
Paper	0.33 (3.14)	1.86	1.46	8.87	–6.15	13.30
Cement	0.49 (1.89)	4.55	–0.93	80.54	–49.81	46.05
Fuel	0.56 (3.53)	2.78	1.70	9.53	–11.44	16.68
Transport	0.36 (1.67)	3.82	1.93	13.77	–17.76	27.27
Insurance	0.34 (3.19)	1.88	0.77	4.82	–6.54	11.17
Micell.	0.21 (2.99)	1.21	0.27	0.81	–3.26	4.42

The Pearson's coefficient of skewness (column 3) indicates that all returns series, except Cement, are positively skewed and statistically significant. The kurtosis coefficient (column 4) varies from 0.81 (implying platykurtic) to 80.54 which reflects a highly leptokurtic distribution. Thus the underlying distributions do not appear to be normal.

We turn now to testing for linear dependence in the data. Table 2 reports autocorrelations of order 1 through 10 for the different stock returns. In addition the Box Pierce Q-statistics for the tenth and lower order of autoregression is also presented. The critical value of this statistic (which is distributed chi-square) is 15.987 at the 5 percent level of significance. The null hypotheses of no first to tenth order autocorrelations are obviously rejected. Except for sugar all first-order autocorrelations are significant at the 5 percent level of significance and the higher order autocorrelations (though significant) start decaying after the third lag. Both the size of the autocorrelations and the Box-Pierce Q-statistic indicate that the returns are linearly dependent.

Table 2

Autocorrelation Coefficients of Log Stock Price Changes
$\log (S_t / S_{t-1}).100$

	Autocorrelations of Order										
Index	1	2	3	4	5	6	7	8	9	10	Q (10)
Gen	.45	.24	.33	.27	.03	–.02	–.03	–.09	–.15	–.21	163
Cot	.41	.22	.20	.16	–.01	–.03	–.02	.01	–.01	–.08	90
Che	.28	.17	.23	.20	.06	.07	.04	.01	–.06	–.08	70
Eng	.28	.18	.16	.14	–.01	.01	–.05	–.11	–.10	–.19	67
Sug	.06	.17	.08	.03	–.11	–.05	–.03	–.06	–.09	–.16	30
Pap	.19	.22	.17	.03	–.02	–.05	–.03	–.08	–.02	–.02	38
Cem	–.30	.08	–.00	.07	–.05	.01	.02	–.04	.02	–.02	34
Fue	.21	.09	.18	.21	–.00	–.12	–.01	–.01	–.08	–.21	62
Tra	.19	–.04	.12	.22	.05	–.08	–.09	–.07–	–.07	–.14	48
Ins	.28	.10	.10	.17	.00	–.09	–.14	–.12	–.11	–.17	67
Mis	.26	.15	.05	.03	–.07	.00	–.07	–.07	–.06	–.02	37

In order to remove the source of linearity from the data we employ a linear model to explain the returns. Assuming efficient markets, we model the actual return at time t, R_t as consisting of the expected return conditional upon the information set at tim t–1, $E\ (R_t \setminus I_{t-1})$, plus a white noise error term, U_t with a finite variance σ_u^2. Formally:

$$R_t = E\ (R_t \setminus I_{t-1}) + u_t = \mu_t + u_t \qquad \text{... ... ...} \qquad (5)$$

and $E\ (u_t \setminus I_{t-1}) = 0$ ... ... (6)

Following the works of Khilji (1993); Rosenberg (1973); Conrad and Kaul (1988) and Koutmos and Lee (1991), we assume that the conditional expected returns (μ_t) are characterised by an error correcting, first-order autoregressive process of the following form:

$$u_t = \mu + \delta\,(\mu_{t-1} - \mu) + V_t \qquad \ldots \quad (7)$$

and $E\,(V_t \backslash I_{t-1}) = 0$

It is assumed that the conditional expected return, μ_t, tends to converge to the long-term (population) mean return, μ. The adjustment factor is δ and v is an error term with mean zero and variance σ_v^2. This is a flexible specification since several models used in the literature are implied by it. If $\delta = 0$, then (7) reduces to the random coefficient model implying that the expected return is constant over time. A value of $\delta > 1$ would imply a nonstationary process while $\delta = 1$ would imply a nonstationary random walk model If μ, the long term rate of return, is zero then (7) becomes an ARMA (1, 1) process.

Since μ are not observed, they need to be estimated. The Kalman filter technique is employed to estimate Equations (5) and (7). Given estimates of the fixed parameters, the Kalman filter recursively updates estimates of the stochastic parameters of the model. The fixed parameters of the model are the long-term mean μ, the adjustment coefficient δ, variance of u, σ_u^2, and the variance of v, σ_v^2. The Berndt, Hall, Hall and Hausman (1974) maximum likelihood estimation technique is used for this to obtain estimates of the fixed parameters.[5]

The estimated long-term expected return was positive and statistically significant at the 20 percent or lower levels of significance for all indices except for transportation. The long term expected return ranged between .19 for miscellaneous and .57 for the fuel index. This implies annual returns of 10.37 percent for miscellaneous and 34.38 percent for the fuel index.

The parameter estimates (not reported) for the adjustment coefficient δ were statistically significant, implying time varying expected returns for most of the indices. These findings were in line with those of Conrad and Kaul (1988) and Koutmos and Lee (1991) who found time varying means for the U.S. and

[5]The reader is referred to Khilji (1993) for details on the estimation of the model. The results are available on request from the author.

other major stock exchanges respectively.[6] All in all the linear model given by Equations (6) and (8) appeared to describe the data satisfactorily.

The residuals from Equation (6) were then checked for any remaining linear dependence. Table 3 gives the autocorrelations for them alongwith the Box-Pierce Q statistic. Note that for six of the indices linear dependence is rejected. These indices are Cotton, Chemicals, Engineering, Paper, Cement, and Transportation. In principle then, it appears that Equations (6) and (8) described these indices adequately and their residuals appear to be generated by random processes. For the other five indices there remains some linear dependence.[7]

Table 3

Autocorrelation Coefficients of Filtered Data

	$\rho1$	$\rho2$	$\rho3$	$\rho4$	$\rho5$	$\rho6$	$\rho7$	$\rho8$	$\rho9$	$\rho10$	Q(10)
Gen	.03	–.16	.16	.18	–.12	–.07	.01	–.04	–.08	–.18	46
Cot	.01	–.06	.05	.11	–.10	–.05	–.02	.04	.01	–.08	12
Chem	.00	–.07	.08	.10	–.05	.01	.00	.01	–.05	–.04	9
Eng	.00	–.03	.04	.08	–.07	.02	–.03	–.08	–.04	–.17	16
Sug	.08	.15	.10	.01	–.09	–.06	–.01	–.08	–.08	–.18	31
Pap	–.05	.07	.08	.08	–.04	–.06	–.06	–.02	–.08	.00	9
Cem	–.00	.00	.04	.07	–.03	.00	.01	–.04	–.00	–.00	3
Fuel	.01	–.09	.08	.17	–.05	–.17	.00	.04	–.02	–.18	34
Tra	–.01	.02	–.01	.03	–.09	.05	–.06	–.05	–.04	–.00	6
Ins	.00	–.04	.02	.17	–.01	–.07	–.10	–.07	–.04	–.12	21
Misc	.26	.15	.05	.03	–.07	.00	–.07	–.07	–.06	–.02	37

In computing the BDS statistics from the filtered data, two important issues have to be dealt with, the choice of ε and the embedding dimension N.

[6]Khilji (1993) found that conditionally expected means of all indices were constant. However he used monthly data, in contrast to weekly data used in this study. The effect of averaging stock prices over a month may have been the removal of the trend in the expected return. The results in this paper seem to corraborate that.

[7]We could have fit a general ARMA model by selecting optimal order of the autoregression for each index. The purpose here was to employ a parsimonious model which had general applicability.

For a given N, ε cannot be too small because $C_N(\varepsilon)$ will capture very few points. On the other hand ε cannot be too large since $C_N(\varepsilon)$ will capture too many points. Similarly N cannot be too large given the relatively small number of observations. For example there would be 154 nonoverlapping 2- histories at dimension 2, but only 77 nonoverlapping 4-histories at dimension 4. As in previous studies, ε is set in terms of the standard deviation of the data, that is $\varepsilon = 1$ means that it is one standard deviation of the data. The BDS statistics are presented for $\varepsilon = 1.5$, 1.25, 1, 0.75, and 0.50 and $N = 2, 3,$ and 4.

Table 4 presents the BDS statistics for the filtered data. Focusing first on the six industries whose residuals appeared to be randomly generated, the BDS statistic clearly rejects iid. Among the other five indices, for which linear dependence was indicated, the hypothesis of iid is accepted for Miscellaneous (surprisingly) but is rejected for the rest as expected since there is still evidence of linear dependence. Generally the BDS statistics decrease for the filtered data.

Table 4

BDS Test: Filtered Data

N	ε	Gen	Cot	Chem	Eng	Sug	Paper	Cem	Fuel	Trans	Ins	Misc
2	1.50	4.04	2.42	3.16	2.02	12.32	1.71	8.59	6.50	12.07	1.60	2.34
3	1.50	3.97	2.13	4.04	1.36	11.11	3.72	7.63	6.12	11.38	3.44	2.60
4	1.50	4.43	2.23	4.63	1.24	10.32	4.11	6.80	5.66	10.51	4.49	2.39
2	1.25	8.11	4.24	4.95	3.65	7.00	1.73	9.00	9.07	10.32	5.22	1.32
3	1.25	8.67	4.54	6.25	3.40	7.09	2.00	7.95	9.23	10.21	6.47	2.23
4	1.25	9.58	5.34	6.96	3.91	7.01	2.37	7.06	9.79	9.86	7.45	2.50
2	1.00	7.34	5.00	5.45	4.30	7.19	2.60	8.39	7.48	7.18	4.91	0.52
3	1.00	8.15	6.23	6.48	4.07	7.87	2.89	8.13	8.29	7.65	6.81	1.38
4	1.00	8.87	7.57	6.91	4.73	8.03	3.53	7.81	8.53	7.80	7.78	2.00
2	.75	5.76	5.16	4.55	4.10	6.92	2.61	9.65	4.59	5.25	3.84	0.50
3	.75	6.52	8.17	5.41	3.77	8.09	2.87	9.81	5.41	5.93	5.61	1.42
4	.75	6.72	10.01	5.82	4.20	8.77	3.48	10.81	5.71	6.51	6.74	2.44
2	.50	5.97	4.53	3.75	4.38	6.56	1.79	9.29	3.47	5.26	3.81	–.26
3	.50	6.57	8.73	4.15	2.68	7.75	1.94	11.05	4.93	6.46	5.82	1.35
4	.50	4.93	12.02	4.09	3.04	10.41	2.97	14.46	6.27	8.00	7.15	2.73

Having ruled out iid for the residuals for Cotton, Chemicals, Engineering, Paper, Cement, and Transportation indices, the BDS test is detecting strong nonlinear dependence in them. Now this could be due to either nonlinear stochastic processes or nonlinear deterministic systems.

4. CONCLUSIONS

This paper has investigated whether weekly stock returns in Pakistan are characterised by linear or nonlinear dependence over the period July 1986 to June 1992. The State Bank of Pakistan's indices of share prices are used to calculate the weekly stock returns for eleven groups of stocks. These consist of an overall (market) index and indices reflecting the stock market performance of ten mutually exclusive industrial groups. The returns of the various series are not normal and are generally positively skewed, leptokurtic, and have a positive mean.

Using an error correcting, first order autoregressive model and employing the Kalman filter estimation technique, we estimated the time varying behaviour of weekly expected returns. Our findings were that the expected monthly returns are time dependent.

The BDS tests were conducted on the residuals from the autoregressive model. Our findings are that six of the indices display strong nonlinear dependence whereas the other five display linear dependence. This nonlinear dependence in the data could result from a nonlinear deterministic (chaotic) system, or a nonlinear stochastic system. In a recent paper Ahmed and Rosser Jr. (1993) report failing to reject the absence of a nonlinear structure in the State Bank of Pakistan's daily stock index even beyond ARCH and exchange rate effects. While our approach is different than their's in that we use weekly data and do not allow for exchange rate and ARCH effects, the similarity in our findings does indicate that future efforts may want to use nonlinear stochastic models like GARCH to estimate the returns.[8] If it turns out that nonlinearity persists in the residuals from these models, then one could conclude that there is low level chaos in some of the data.

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