Algebraic representation of Social Capital Matrix

- Tariq Shah[1], Syed Akhter Hussain Shah[2] and Eatzaz Ahmed[3]
- [1]Department of Mathematics, Quaid-i-Azam University Islamabad,
- [2]Small Industries Development Board, Government of N.-W.F.P.,
- [3]Department of Economics, Quaid-i-Azam University Islamabad.

What is Social Capital?

- In a society, the trust, reciprocity, norms and networks of civic engagement facilitate in working together to achieve desired goals, causes social capital, which is interactive in nature and embedded in the structure of a society.
- Dayton-Johnson state it as: Social capital is an individual's sacrifices (time, effort, consumption) made in an effort to promote cooperation with others and social cohesion is a characteristic of society which depends on the accumulated social capital.

Types of social capital

Structural Social Capital, refers to relatively objective and externally observable social structures, such as networks, associations and institutions and the roles, rules and procedures they embody.

Cognitive Social Capital comprises of more subjective and intangible elements such as generally accepted attitude and norms of behavior, shared values, reciprocity and trust.

The cognitive social capital may create coherence and homogeneity in the group of people having more commonality in their norms, values, behavior, beliefs, reciprocity and attitudes.

Different approaches regarding social capital

- i). Access to information.
- ii. A degree of trust, an expectation of reciprocity and exchange of information are expected to prevail in relationships (social capital).
- iii). Putnam states it is a precondition for economic development as well as for effective government.

- v). People trust each other and cooperate more for common causes.
- vi). Robinson and Flora are of the view that individual utility-maximizing behavior cannot be pursued independent of the wellbeing of others.
- vii). Cox contends that individuals' lives are about their relationships with others, but involve levels of trust and cooperation or anger and distrust. These comprise individuals' social capital, which makes democracy work, production rise and societies cohesian develop.
- viii). Grootaert and Bastelaer view social capital as "the institutions relationship, attitudes, and values that govern interaction among people and contribute to economic and social development".
- ix). Shah. Akhter (2007) points out that individuals maintain their social interactions on the basis of their actual and expected returns (welfare) from their relationships with others.

Motivations behind this study

Social Capital Matrix

- Social capital may exist in following interactive forms:
- individual vs. individual, individual vs. group or community, individual vs. institution or organization, individual vs. state,
- group or community vs. group or community, group or community vs. institution or organization, group or community vs. state,
- institution or organization vs. institution or organization, institution or organization vs. state
- state vs. state.
- Accumulation of social capital in different dimensions is reflected in a matrix form (here we must indicate that it does not fulfill the complete sense of a matrix as in algebra), we call Interactive social capital matrix(abrivatd. LCOS), as shown in the following table and which has 16 different entries.

Social Capital Matrix 16-dimensions

Stake holders	*L*	*C*	*O*	*S*
L	*L* vs *L*	*L* vs *C*	*L* vs *O*	*L* vs *S*
C	*C* vs *L*	*C* vs *C*	*C* vs *O*	*C* vs *S*
O	*O* vs *L*	*O* vs *C*	*O* vs *O*	*O* vs *S*
S	*S* vs *L*	*S* vs *C*	*S* vs *O*	*S* vs *S*

Algebraic structures, representatives of systems in Social Capital Matrix

- Social capital has wide range, number of dimensions, therefore need to be coded for analysis. The special algebraic structures are used to codify the concepts, type and mode of transaction of social capital amongst different players. We selected algebraic structures of particular interest, that is, finite fields, vector spaces, homomorphisms of rings and linear transformation.

Continue

- Group: G≠Ø with an associative binary operation * (i.e.,a* b∈G,(a* b)* c= a* (b* c) for a,b,c∈G) is a group if e* r=r* e=r, we call e, the identity element in G and for each g∈G, there exist h∈G such that g* h=h* g=e, whereas we call g and h, the inverses of each other. Group G is commutative if a* b=b* a,a,b∈G.

  Ring: A commutative group (R,+) is a ring if in R, "." is associative binary operation and "." is distributive over "+". A ring R is commutative if a.b=b.a, for all a,b∈R. A ring R is with identity if 1∈R. Let R be a commutative ring with identity. a∈R is said to be invertible or unit in R if there exist b∈R such that ab=ba=1. U(R), the set of all unit elements in R. (U(R),.) forms a group.

continue

- Field: A commutative ring R with 1is said to be an integral domain if ab=0, where $a,b \in R$, then either a=0 or b=0. A commutative ring F with identity is said to be field if U(F)=F\{0}. A finite integral domain is a field but every field is not an integral domain.

continue

- Finite field: Given nonnegative integers 0<a and b, there exist q=0 and r with 0=r=a such that b=aq+r, where q is quotient and r is remainder which are unique (Division algorithm is stated). If r=0, we say a divides b (that is a| b).

- For a fixed positive integer m, we say $a,b \in Z$ are congruent modulo m, written $a=b \pmod{m}$ if $m \mid a-b$ or equivalently, if $a=b+mt$, where $t \in Z$. Here m is called the modulus (plural; moduli). $a=0 \pmod{m}$ means $m \mid a$, $a=b \pmod{1}$ for all $a,b \in Z$, therefore we consider the positive integer $m>1$ and $\{b+mt : t \in Z\}$ is the set of integers to which b is congruent modulo m.

- Every integer is congruent modulo m to exactly one of the numbers in the set {0,1,2,..,m-1}. Let 2=m be the modulus, which is fixed. Define the congruence class of b (mod m), written [b]_{m} as

$$[b]_m = \{a \in \mathbb{Z} : a \equiv b \pmod{m}\} = \{a \in \mathbb{Z} : m \text{ divides } a - b\}$$

$$= \{a \in \mathbb{Z} : a = b + mt, \text{ where } t \in \mathbb{Z}\}.$$

$[a]_m = [b]_m$ if and only if $a \equiv b \pmod{m}$

continue

Every congruence class mod m is equal to one of $[0]_m, [1]_m, [2]_m, \ldots, [m-1]_m$. Obviously all these classes are different. Thus there are only m congruence classes for modulo m. We represent the set of all congruence classes modulo m by $\mathbb{Z}_m$. So

$$\mathbb{Z}_2 = \{[0]_2, [1]_2\}, \mathbb{Z}_3 = \{[0]_3, [1]_3, [2]_3\}, \ldots \mathbb{Z}_m = \{[0]_m, [1]_m, [2]_m, \ldots, [m-1]_m\}.$$

In $\mathbb{Z}_m$ we define the binary operations $\oplus_m$ and $\odot_m$. If $n = 2$, then $\mathbb{Z}_2 = \{[0]_2, [1]_2\}$ and we define the binary operations $\oplus_2$ and $\odot_2$ as follow

$\oplus_2$	$[0]$	$[1]$
$[0]$	$[0]$	$[1]$
$[1]$	$[1]$	$[0]$

and

$\odot_2$	$[0]$	$[1]$
$[0]$	$[0]$	$[0]$
$[1]$	$[0]$	$[1]$

.

$\mathbb{Z}_m$ is an integral domain (and hence a field) if and only if m is prime integer. For example $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5, \ldots$ are finite fields with $2, 3, 5, \ldots$ elements (vectors).

Vector space

- An additive abelian group V is said to be a vector space or linear space over the field F if the scalar multiplication map $F \times V \rightarrow V$, defined as $(a,v) \mapsto av$ satisfies
- (i) $a(v+w)=av+aw$;
- (ii) $(a+ß)v=av+ßv$;
- (iii) $(aß)v= a(ßv)$;
- (iv) $1.v=v$, for all $a,ß \in F$, $v,w \in V$.

Algebra

- A vector space V is said to be an algebra over the field F if V is ring and a(vw)= (av)w=v(aw) for all $a \in F$, $v, w \in V$.
- A field is not only a vector space over itself with dimension 1but it is an example of algebra.Furthermore for a positive integer n,

$F^n = \{(a_1, a_2, \dots, a_n) : a_1, a_2, \dots, a_n \in F\}$ is an algebra over F with dimension n. If p is prime integer and n be any positive integer, then $\mathbb{Z}_p$ is a one dimensional algebra over the field $\mathbb{Z}_p$ and $\mathbb{Z}_p^n$ is n dimensional algebra over the field $\mathbb{Z}_p$, particularly we may take $p = 2$. Interestingly $\mathbb{Z}_2$ is a *Boolean algebra*, as $a^2 = a$ and $a + a = 0$, for all $a \in \mathbb{Z}_2$.

Ring homomorphism

- 1. Let R and S be commutative rings. A ring homomorphism is a map φ: R→S if for all x, y∈R, φ(x+y)=φ(x)+φ(y) and φ(xy)=φ(x)φ(y). A ring homomorphism φ is said to be a monomorphism (resp. epimorphism, isomorphism) if φ is one-one (resp. onto, bijective).

Linear transformation

- Let V and W be finite dimensional vector spaces over the same field F. A vector space homomorphism (linear transformation) is a map $\varphi: V \rightarrow W$ which satisfies $\varphi(x+y)=\varphi(x)+\varphi(y)$ and $\varphi(ay)=a\varphi(y)$, for all $x, y \in V, a \in F$. A vector space homomorphism is an isomorphism if it is bijective. If φ is isomorphism, then we say V is isomorphic to W and we represent it as $V \cong W$.

Algebraic Representation of Social Capital Matrix

- The algebraic representation is devised in view of the four types of players (systems) namely State, organization, community and individuals and their interactions as described in social capital matrix but we assumed that behind these interactions the economics of spending and welfare work, ultimately which cause to establish the respective economic and social environment.

- The new environment emerge to accumulate social capital and economic development amongst different categories in respective system. By different interactions we obtain the 4×4 matrix, which contains 16 different intera active and interactive environments, known as social capital matrix.

The model (adjustments)

- The state S is labelled with Boolean algebra $Z2=\{0,1\}$, which have two active categories (vectors) 0,1denoted as S-vectors or S-categories. Furthermore the category 0 represents the investments/spending and 1 represents the return/welfare indicator of the state.

- We assume that linear space Z_2^2 represents organization O with 4 O-vectors or O-categories (of organizations). Likewise Z_2^3 and Z_2^4 represent community C with 8 C-vectors or C-categories (of communities) and individual L with 16 L-vectors or L-categories (of individuals) respectively

$\mathbb{Z}_2 \downarrow$ **State** (S)

$\mathbb{Z}_2^2 \downarrow$ **Institution/Organization** (O)

$\mathbb{Z}_2^3 \downarrow$ **Group/Community** (C)

$\mathbb{Z}_2^4 \downarrow$ **Individual** (L).

- This formation will lead to the format of Social Capital Matrix, that is State-Organization-Community and then individual (abbreviated as SOCL), which may be interpreted as the state leading the all types of activities through organization, community and finally, individual.

- Obviously, this format can be criticized on the basis of the presumption that the individuals constitute the communities, the communities constitute the organization and the organization constitute the state, which would require the reverse format individual-Community-Organization and then State (abbreviated as LCOS) in Ph. D thesis of second author. But in our case, the business of a state depending on two indicators, running all other systems by its authoritative position.

- So it is essential that we have to consider the reverse order SOCL for social capital matrix LCOS.

$\rtimes$	*S*	*O*	*C*	*L*		$\rtimes$	Z_2	Z_2^2	Z_2^3	Z_2^4
S	*SS*	*SO*	*SC*	*SL*		Z_2	$Z_2 \rtimes Z_2$	$Z_2 \rtimes Z_2^2$	$Z_2 \rtimes Z_2^3$	$Z_2 \rtimes Z_2^4$
O	*OS*	*OO*	*OC*	*OL*	✵	Z_2^2	$Z_2^2 \rtimes Z_2$	$Z_2^2 \rtimes Z_2^2$	$Z_2^2 \rtimes Z_2^3$	$Z_2^2 \rtimes Z_2^4$
C	*CS*	*CO*	*CC*	*CL*		Z_2^3	$Z_2^3 \rtimes Z_2$	$Z_2^3 \rtimes Z_2^2$	$Z_2^3 \rtimes Z_2^3$	$Z_2^3 \rtimes Z_2^4$
L	*LS*	*LO*	*LC*	*LL*		Z_2^4	$Z_2^4 \rtimes Z_2$	$Z_2^4 \rtimes Z_2^2$	$Z_2^4 \rtimes Z_2^3$	$Z_2^4 \rtimes Z_2^4$

Components in Categories of the Systems

- On the basis of size of the systems we may call S is smaller than O, O is smaller than C and C is smaller than L or L is larger than C. As

$$Z_2^m = \{(a_1, a_2, .., a_m) = a_1 a_2 .. a_m : a_1, a_2, .., a_m \in Z_2\} \text{ and}$$

$$Z_2^l = \{(a_1, a_2, .., a_l) = a_1 a_2 .. a_l : a_1, a_2, .., a_l \in Z_2\}$$

are m and $l-$dimensional linear spaces over the field $\mathbb{Z}_2$ respectively. So $\mathbb{Z}_2^l \times \mathbb{Z}_2^m \cong \mathbb{Z}_2^{l+m}$ is $l+m$ dimensional linear space over the field $\mathbb{Z}_2$. Whereas in this study $1 \le l, m \le 4$. So the interaction of any two systems can be represented as like S, O, C and L.

The System V	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2 \times \mathbb{Z}_2^4$..
No. of categories of V	2	4	8	16	32..
No. of components in the categories of V	1	2	3	4	5..

- In this work by economic development (respectively social capital) we mean the economic development (respectively social capital) in the respective categories of organization, community and individual. This means we may classify the organizations, communities and individuals into different categories regarding economic development (respectively social capital).

The top row and first column of SOCL

- The state Z_2, Organization Z_2^2, Community Z_2^3 and Individual Z_2^4 have 2,4,8 and 16 categories respectively, which are representing the investments/spending and return/welfare indicator.

	Z_2	Z_2^2	Z_2^3	Z_2^4
Z_2				
Z_2^2				
Z_2^3				
Z_2^4				

Intra action of the systems (The main diagonal of SOCL)

These are the interactions of a system with itself, i.e. State vs. State, Organization vs. Organization, Community vs. Community, Individual vs. Individual. We may call all of 4, the Intra action of the systems, that is these activities are on main diagonal of SOCL.

$$Z_2 \lhd Z_2$$

$$Z_2^2 \lhd Z_2^2$$

$$Z_2^3 \lhd Z_2^3$$

$$Z_2^4 \lhd Z_2^4$$

Intra-active function

$$f : Z_2^m \times Z_2^m \rightarrow Z_2^m, \text{ where } 1 \le m \le 4$$

$$f((a_1..a_m),(b_1..b_m)) = (c_1..c_m) \in Z_2^m, (a_1..a_m),(b_1..b_m) \in Z_2^m,$$

- whereas ci=ai+bi, 1= i=4. We call d intra-active function, which is interpreted as the economic trade off among the categories of one of the four systems. However in resulting one can obtain again a category of the same system. A category has m! number of possibilities regarding its status in respect of economic development or accumulation of social capital of categories of the systems:

- For instance if $m=3$, then $d: Z_2^3 \times Z_2^3 \rightarrow Z_2^3$, defined as $d(a_1a_2a_3, b_1b_2b_3)=c_1c_2c_3 \in Z_2^3$, where $c_i=a_i+b_i$. $1=i=3$

- Since $Z2^3$ represent the community, so d explains the community vs. community. In this type of interaction all components of two categories of the community is doing business with all of their corresponding components. This also reflects that the total assets of interactive categories of the community are fully operationalized and no part left for substance for its own survival.

- Hence this indicates the case, that is in favour to this finding that categories of the community that consumes/spend all of its assets/resources in one period. This also indicates that intra-action of any system provide a high level of trust among the categories of the same system, which causes in economic development and accumulation of social capital of categories and hence to the system under consideration.

Across interactions (Lower and Upper Diagonal of SOCL)

- 1. The interaction of State & Individual, has the representation $Z2\times Z24$ (resp. the interaction of Individual & State has the representation $Z24\times Z2$).
- 2. The interaction of State & Community has the representation $Z2\times Z2^3$ (resp. the interaction of Community & State has the representation $Z2^3\times Z2$).

- 3. The interaction of State & Organization has the representation $Z_2 \times Z_2^2$ (resp. the interaction of Organization & State has the representation $Z_2^2 \times Z_2$).
- 4. The interaction of Organization & Individual has the representation $Z_2^2 \times Z_{24}$ (resp. the interaction of Individual & Organization has the representation $Z_{24} \times Z_2^2$).

- 5. The interaction of Organization & Community has the representation $Z_2^2 \times Z_2^3$ (resp.the interaction of Community with Organization has the represtn. $Z_2^3 \times Z_2^2$). 6. The interaction of Community & Individual has the representation $Z_2^3 \times Z_2^4$ (resp. the interaction of Individual & Community has the representation $Z_2^4 \times Z_2^3$).

- These are representing 12 numbers of across interactions of the systems, i.e. State vs. Organization and vice versa, State vs. Community and vice versa, Community vs. Individual and vice versa. We may call these Lower and Upper Diagonal Interactions (LUD-Interactions), that is these are not on the main diagonal of SOCL. It may be represented as

	$Z_2 \lhd Z_2^2$	$Z_2 \lhd Z_2^3$	$Z_2 \lhd Z_2^4$
$Z_2^2 \lhd Z_2$		$Z_2^2 \lhd Z_2^3$	$Z_2^2 \lhd Z_2^4$
$Z_2^3 \lhd Z_2$	$Z_2^3 \lhd Z_2^2$		$Z_2^3 \lhd Z_2^4$
$Z_2^4 \lhd Z_2$	$Z_2^4 \lhd Z_2^2$	$Z_2^4 \lhd Z_2^3$	

Lower Diagonal and Upper Diagonal interactions having symmetries due to this model, because $Z_2^l \triangleleft Z_2^m$ and $Z_2^m \triangleleft Z_2^l$, $1 \leq l, m \leq 4$, are same in nature in algebraic perspective(i.e., isomorphic). But in interactive as in the social capital matrix they are dealing their affaires differently.

First it is noticed that if $0 = \{0\}$ is zero vector space, consisting on 0 vector only. So, for $l < m$, $Z_2^l \to Z_2^m$ is imbedding of Z_2^l in Z_2^m, i.e. $Z_2^l \to Z_2^l \oplus 0 \oplus . \oplus 0 \to Z_2^m$, this means $a_1..a_l = a_1..a_l 0_{l+1}..0_m \in Z_2^m$. Similarly $m < l$, $Z_2^m \to Z_2^l$ is imbedding of Z_2^m in Z_2^l, i.e. $Z_2^m \to 0_1 \oplus . \oplus 0_{l-m} \oplus Z_2^m \to Z_2^l$, this means $a_1..a_l = 0_1..0_{l-m} a_1..a_l \in Z_2^l$.

Now we define functions $N_{m,l}$ and $N_{l,m}$ as follow:

$$N_{l,m} : Z_2^l \times Z_2^m \to Z_2^m, \text{ where } 1 \le m \le 4, \text{ and } m \le l$$

by $N_{l,m}(a_1..a_l, b_1..b_m b_{m+1}..b_l) = c_1..c_m c_{m+1}..c_l \in Z_2^m$

for any $a_1..a_l \in Z_2^l, b_1..b_m \in Z_2^m$ and $b_{m+1} = \ldots = b_l = 0$.

and

$$N_{l,m} : Z_2^l \times Z_2^m \to Z_2^m, \text{ where } 1 \le m \le 4, \text{ and } l \le m$$

by $N_{l,m}(a_1..a_l a_{l+1}..a_m, b_1..b_m) = c_1..c_l c_{l+1}..c_m \in Z_2^m$,

for any $a_1..a_l \in Z_2^l, b_1..b_m \in Z_2^m$ and $a_{l+1} = \ldots = a_m = 0$

Whereas c_i ■ a_i □ b_i, 1 ✧ i ✧ 4. We call ☠$_{m\&l}$ and ☠$_{l\&m}$, the across inter-active functions, which are interpreted as the economic trade off among the categories of different systems. However in result of this trade off, again a category is obtained, which is infact belongs to larger systems of across inter-active systems.

By across inter-active function ☠$_{l \otimes m}$, l ✧ m we conclude that interaction of the system Z_2^l with l ⊟1, l ⊟2, .., m components of the larger system (in size and dimension) Z_2^m remains inactive dur only first l number of components interact with their corresponding l members in the smaller systen Z_2^l. Similarly by across inter-active function ☠$_{l \otimes m}$, m ✧ l we conclude that interaction of the syste that the 1, 2, .., m components of the larger system (in size and dimension) Z_2^l remains inactive du only last m number of components interact with their corresponding m members in the smaller syst dimension) Z_2^m.

This can be illustrated by an example, for instance if we consider $\phi_{23} : Z_2^2 \times Z_2^3 \to Z_2^3$, then

$$\phi_{23}(a_1a_20, b_1b_2b_3) = c_1c_2c_3 \in Z_2^3, \text{ where } c_3 = b_3.$$

- By across inter-active function $d_{m \hookrightarrow l}$, $m=l$ we conclude that interaction of the system $Z2^{m}$ with $Z2^{l}$ provided that the $m+1, l+2, .., l$ components of the larger system (in size and dimension) $Z2^{l}$ remains inactive during interaction, i.e. the only first m number of components interact with their corresponding m members in the smaller system (in size and dimension) $Z2^{m}$. Now it can be interpreted if we consider $d_{3 \hookleftarrow 2}: Z2^3 \times Z2^2 \rightarrow Z2^3$, then

- $d_{3\leftarrow 2}(a_1a_2a_3, 0b_2b_3)=c_1c_2c_3 \in Z_2^3$, where $c_1=a_1$. Recall that Z_2^2 represent the organization and Z_2^3 represent the community. The $d_{2\rightarrow 3}$ explains the organization vs. community. In organization vs. community the first two components of category of the community are doing business with all two components of the organization.

- This means the total assets are not operationalized by the community rather a part left for substance for its own survival. This also reflects extreme case, that is in contradiction to this finding that community or organization that consumes/spend all of its assets/resources in one period do not survive for next period. Furthermore it is not like the intra-action of a system. In similar fashion if $d_{3\hookrightarrow 2}:Z2^{3}\times Z2^{2}\to Z2^{3}$,

- As $Z2^2$ represent the organization and $Z2^3$ represent the community. The d_{3↩2} explains the community vs. organization. In community vs. organization the first two components of category of the community are doing business with all two components of the organization. This also reflects that the total assets are not operationalized by the community rather a part left for substance for its own survival

- These findings strengthened our format of Social Capital Matrix, that is SOCL, which compels for the pivital role of state in all types of activities of categories of organization, community and finally, individual.
- The following represent the SOCL

L

←

C

←

O

←

S

Conclusions

1. The social capital matrix, emerges through interaction of State, Organization, Community and the individual, we identified as system S, the system O, the system C and the system L respectively. Through algebraic representation of social capital matrix with assumption that S,O,C and L have 2, 4, 8 and 16 categories respectively, we have found that the economic development and hence social capital in each category of any system can be determine.

2. Further we have observed that in each category (consisting on 4 economic indicators) of individual there is a reflection of the presence of 3 economic indicators of a category of the community. As well in each category of community there is a reflection of 2 economic indicators of a category of organization. Similarly in the each category of organization there is a reflection of 1economic indicator of the state.

3. Across interaction of different systems provided that not all the components of a category of the larger system are doing business with the components of the smaller system. This shows that the total assets/resources are not operationalized by the larger system rather a part left for substance for its own survival. This also reflects extreme case, that is in contradiction to this finding that community or organization that consumes/spend all of its assets/resources in one period do not survive for next period.

4. During intra-action of a system all components of two interactive categories are doing business with all of their corresponding components, which reflects that the total assets of interactive categories of the system under consideration are fully operationalized and no part left for substance for its own survival.

5. This also indicates that intra-action of any system provides a high level of trust among the categories of the same system, which causes economic development and accumulation of social capital of categories and hence to the system concern. It can also be observe that it is not like across interaction of different systems

Thanks for patience